EFFICIENT NUMERICAL SOLUTION OF MULTI-ORDER FRACTIONAL INTEGRO DIFFERENTIAL EQUATIONS.

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Abstract
This work provides numerical solution to multi-order fractional Integro differential equations by Standard Collocation Method (SCM) using power series and Legendre polynomial bases functions. The method assumed an approximate solution of the forms of power series and Legendre polynomials bases functions. The assumed approximate solution is now substituted into the multi order fractional integro differential equation. After a careful implementation of the fractional order differential and integral operators, we compare coefficients of powers of the variable to obtain a system of algebraic equations which are solved using the successive backward substitution method. Comparison of performance of the bases functions used was also carried out. The results revealed that the method is simple and efficient. The results converge to the exact solutions after little iterations. Numerical examples are given to illustrate the method.

Keywords: Basis functions, Power series and Legendre polynomial

1 Introduction

Most Scientific and Engineering problems are better presented and solved using fractional calculus. Khader (2011) stated that owing to the growing applications of fractional calculus in the formulation and representation of physical life problems, it may soon become the calculus of the twenty-first century. This statement speaks volume of the fact that the applications of fractional calculus in the representation and presentation of life problems cannot be over emphasized. On the other hand, the usefulness and applications of fractional integro differential equations has grown tremendously so much so that a lot of researchers have picked interest in them. One fact however remains that most fractional integro differential equations cannot be solved analytically. So approximation and numerical techniques must be developed and used to solve them. To this end, several numerical methods have been advanced. They includes and not limited to; Laplace Transformation Method (LTM), Adomian Decomposition Method (ADM), Variational Iterative Method (VIM), Homotopy Analysis Method.
(HAM), Galerkin Method (GM), Collocation Method (CM), Successive Substitution Method (SSM), and so on.

Taiwo and Odetunde (2013) used Iterative Decomposition Method for the approximation of multi order fractional differential equations. The paper presented solutions of rapidly convergent series of computable terms. Ibtisam (2011) applied the Homotopy Analysis Method to solve multi order fractional order integro differential equations involving Initial value problems. Khader (2011) carried out implementation of operational matrix of fractional derivative to solve nonlinear multi order fractional differential equations using Legendre polynomial. Aatabakzadeh, Arami and Erjaee (2012) used Chebyshev operational matrix method to solving multi order fractional ordinary differential equations. By using shifted Chebyshev polynomial, the authors were able to obtain a satisfactory result without linearization, discretization or perturbation. Davood, Moshsen and Dumitru (2013) solved multi term orders fractional differential equations by operational matrix of BPs with convergence analysis. The method involves using matrices to reduce linear multi-term order fractional differential equations to a system of algebraic equations. In this paper, we implement Legendre polynomial and Power Series to solve multi-order fractional integro differential equations using collocation method. It involves using a systematic successive backward substitution method to get the approximate solution.

2 Basic Definitions to the work

Here, we present some basic definitions of fractional calculus which are very useful in this work.

2.1 Definition

Fractional derivation, $D^\alpha_x f(t)$ in the Caputo sense is given as

$$D^\alpha_x f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^n(\tau) \, d\tau. \quad (1)$$

for $n-1 < \alpha \leq n, \quad n \in \mathbb{N}, \quad t > 0, \quad f(t) \in C^\mu, \quad \mu > 0.$

where $f(t)$ is a function to be determined, $\alpha$ is the order of derivative of the problem and $n$ is the degree of order of the approximation of the function.

2.2 Definition 2

The general form of multi order fractional integro differential equation is given as

$$D^\alpha_x u(t) = f(x) + J^\beta(k(x,t))u(t) \quad (2)$$

*
$D^\alpha$ and $J^\beta$ are the Caputo sense of the differential and integral of the given functions respectively $\alpha$ and $\beta$ are parameters denoting the fractional order derivative and integral of the functions. Some important properties of $D^\alpha f(t)$ and $J^\beta f(t)$ are:

$$D^\alpha t^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha}$$

and

$$J^\beta t^n = \frac{\Gamma(n+1)}{\Gamma(n+\beta+1)} t^{n+\beta}$$

(3)

3. Methodology

3.1 Power Series Basis Function

Consider a multi order fractional integro differential equation of the form (2). We assumed an approximate solution in power series of the form

$$u_N(t) = \sum_{i=0}^{N} a_i t^i$$

(4)

where $a_i, \ i = 0(1)N$ are constants to be determined and $t^i, \ i = 0(1)N$ are the power series terms. Substituting (4) into (2), we have

$$D^\alpha \left\{ \sum_{i=0}^{N} u_N(t) \right\} = f(x) + k(x,t)J^\beta \left\{ \sum_{i=0}^{N} u(t) \right\}$$

(5)

The fractional differential operator $D^\alpha$ is applied to the left hand side (LHS) while the integral operator $J^\beta$ is applied to the right hand side.

then, (7) becomes

$$\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \left\{ \sum_{i=0}^{N} u_N(t) \right\} = f(t) + \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} \left\{ \sum_{i=0}^{N} u_N(t) \right\}$$

(6)

we expand and implement the $D^\alpha$ and $J^\beta$, we have the form:
\[ D^\alpha u_0 + D^\alpha u_1(t) + \cdots + D^\alpha u_N(t^N) = f(x) + J^\beta u_0 + J^\beta u_1(t) + \cdots + J^\beta u_N(t^N) \]  

(7)

Now, we rearrange (7) to have

\[ D^\alpha u_0 + D^\alpha u_1(t) + \cdots + D^\alpha u_N(t^N) - \left\{ J^\beta u_0 + J^\beta u_1(t) + \cdots + J^\beta u_N(t^N) \right\} = f(x) \]  

(8)

and (8) leads to a system of equations from which we obtain the values of the unknowns by comparing coefficients of powers of \( t \).

The values, \( a_i \) (\( i = 0(1)N \)) are put back into the assumed approximate solution using a systematic successive backward substitution method to get the approximate solution.

### 3.2 Legendre Basis Function

Here we follow the same procedure as above but using the Legendre polynomial basis function. We assume an approximate solution of the form

\[ y_N(t) = \sum_{i=0}^{N} c_i P_i(t) \]  

(9)

where \( c_i, \ i = 0(1)N \) are constants to be determined and \( P_i(t) \) are the Legendre polynomials Substituting (9) into (2) and with some further simplifications we have

\[ D^\alpha \left\{ \sum_{i=0}^{N} y_N(t) \right\} = f(x) + k(x,t)J^\beta \left\{ \sum_{i=0}^{N} y(t) \right\} \]  

(10)

Expanding (10) and applying the differential and integral operators, we have

\[ D^\alpha y_0(t) + D^\alpha y_1(t) + \cdots + D^\alpha y_N(t) - J^\beta y_0(t) - J^\beta y_1(t) - \cdots - J^\beta y_N(t) = f(t) \]  

(11)

Now we rearrange (11), we have

\[ D^\alpha y_0(t) + D^\alpha y_1(t) + \cdots + D^\alpha y_N(t) - \left\{ J^\beta y_0(t) + J^\beta y_1(t) + \cdots + J^\beta y_N(t) \right\} = f(t) \]  

(12)

Hence, from equation (12) we now obtain the value of the unknown constants by equating the corresponding powers of \( t \).

### 4 Numerical Examples

**Example 1:**

Consider the initial value problem that consists of the multi-fractional order nonlinear
**integro differential equation.**

\[ D^{0.5}y(t) = \frac{6}{\Gamma(3.5)}t^{2.5} - \frac{\Gamma(4)}{\Gamma(5.5)}t^{4.5} + J^{1.5}y(t) \quad t \in [0, 1] \quad (13) \]

Subject to \( y(0) = 0 \). The exact solution is \( y(t) = t^3 \).

We use power series method to solve this IVP of multi-order fractional problem \( N = 4 \) by assuming an approximate solution of the form:

\[ y_4(t) = \sum_{i=0}^{4} a_i t^i = a_0 t^0 + a_1 t^1 + a_2 t^2 + a_3 t^3 + a_4 t^4 \quad (14) \]

Applying (3) to the (14), we have

\[ D^{0.5}y_n(t) = D^{0.5}\{a_0 t^0 + a_1 t^1 + a_2 t^2 + a_3 t^3 + a_4 t^4\} \quad (15) \]

\[ = a_0 \frac{\Gamma_1}{\Gamma_3} t^{-\frac{1}{2}} + a_1 \frac{\Gamma_2}{\Gamma_3} t^{\frac{1}{2}} + a_2 \frac{\Gamma_3}{\Gamma_5} t^{3} + a_3 \frac{\Gamma_4}{\Gamma_7} t^{5} + a_4 \frac{\Gamma_5}{\Gamma_9} t^{7} \quad (16) \]

Also,

\[ J^{1.5}y(t) = J^{1.5}\{a_0 t^0 + a_1 t^1 + a_2 t^2 + a_3 t^3 + a_4 t^4\} \]

\[ = a_0 \frac{\Gamma_1}{\Gamma_5} t^{\frac{5}{2}} + a_1 \frac{\Gamma_2}{\Gamma_5} t^{\frac{7}{2}} + a_2 \frac{\Gamma_3}{\Gamma_7} t^{9} + a_3 \frac{\Gamma_4}{\Gamma_9} t^{11} + a_4 \frac{\Gamma_5}{\Gamma_11} t^{13} \quad (17) \]

we rewrite (12) as

\[ D^{0.5}y(t) - J^{1.5}y(t) = \frac{6}{\Gamma_5} t^{\frac{5}{2}} - \frac{\Gamma_4}{\Gamma_9} t^{\frac{9}{2}} \quad (18) \]

and substituting (16) and (17) in (18), we have

\[ a_0 \left[ \frac{\Gamma_1}{\Gamma_3} t^{-\frac{1}{2}} - \frac{\Gamma_1}{\Gamma_5} t^{\frac{5}{2}} \right] + a_1 \left[ \frac{\Gamma_2}{\Gamma_3} t^{\frac{1}{2}} - \frac{\Gamma_2}{\Gamma_5} t^{\frac{7}{2}} \right] + a_2 \left[ \frac{\Gamma_3}{\Gamma_5} t^{3} - \frac{\Gamma_3}{\Gamma_7} t^{9} \right] + a_3 \left[ \frac{\Gamma_4}{\Gamma_7} t^{5} - \frac{\Gamma_4}{\Gamma_9} t^{11} \right] + a_4 \left[ \frac{\Gamma_5}{\Gamma_9} t^{7} - \frac{\Gamma_5}{\Gamma_11} t^{13} \right] = \quad (19) \]

Comparing coefficients of the powers of t of (19), we have:
Substituting these values of into assumed approximate solution, we have

\[ y_4(t) = a_0 t^0 + a_1 t^1 + a_2 t^2 + a_3 t^3 + a_4 t^4 \]

\[ = 0 + 0 + 0 + t^3 + 0 = t^3 \]

\[ y_4(t) = t^3 \text{ which is the exact solution.} \]

**Example 2:**
Consider the initial value problem that consists of the multi-fractional order nonlinear integro differential equation

\[ D^{0.75}y(t) = \frac{1}{\Gamma(1.25)}t^{0.25} - \frac{2}{\Gamma(3.75)}t^{3.75} + \int^{0.75}y^2(t) \quad t \in [0, 1] \]  

(22)

Together with the initial condition; \( y(0) = 0 \)

The exact solution is \( y(t) = t \)

We use Legendre polynomial basis function to solve this IVP of multi-order fractional problem for \( N = 2 \) by assuming an approximate solution of the form:

\[ y_2(t) = \sum_{i=0}^{2} c_i P_i = c_0 P_0 + c_1 P_1 + c_2 P_2 \]  

(23)
Where \( c_i, \ i = 0(1)N \) and \( P_i \) are Legendre polynomials

\[
D^\frac{3}{4}\{y_2(t)\} = D^\frac{3}{4}\{c_0 P_0\} + D^\frac{3}{4}\{c_1 P_1\} + D^\frac{3}{4}\{c_2 P_2\} = \frac{\Gamma 1}{\Gamma^\frac{1}{4}} t^\frac{-3}{4} + c_1 \frac{\Gamma 2}{\Gamma^\frac{1}{4}} t^\frac{1}{4} + c_2 \left[ \frac{\Gamma 3}{\Gamma^\frac{9}{4}} t^\frac{5}{4} - \frac{\Gamma 1}{\Gamma^\frac{1}{4}} t^\frac{3}{4} \right] \quad (24)
\]

\[
J^\frac{3}{2}\{y_2^2(t)\} = J^\frac{3}{2} \left\{ \left( c_0^2 + \frac{1}{4} c_2^2 - c_0 c_2 \right) + (2c_0 c_1 - c_1 c_2) t + \left( 3c_0 c_2 - \frac{3}{2} c_2^2 + c_1^2 \right) t^2 + 3c_1 c_2^2 t^3 + \frac{9}{4} c_2^4 t^4 \right\}
\]

\[
= \left( c_0^2 + \frac{1}{4} c_2^2 - c_0 c_2 \right) \frac{\Gamma 1}{\Gamma^\frac{1}{4}} t^\frac{3}{4} + (2c_0 c_1 - c_1 c_2) \frac{\Gamma 2}{\Gamma^\frac{11}{4}} t^\frac{7}{4} + \left( 3c_0 c_2 - \frac{3}{2} c_2^2 + c_1^2 \right) \frac{\Gamma 3}{\Gamma^\frac{15}{4}} t^\frac{11}{4} + 3c_1 c_2^2 \frac{\Gamma 4}{\Gamma^\frac{19}{4}} t^\frac{15}{4} + \frac{9}{4} c_2^4 \frac{\Gamma 5}{\Gamma^\frac{23}{4}} t^\frac{19}{4} \quad (25)
\]

Now we arrange (22) as \( D^\alpha y(t) - J^\beta y(t) = f(t) \) and substituting (24) and (25) we have

\[
\frac{c_0^2}{\Gamma^\frac{1}{4}} t^\frac{-3}{4} + c_1 \frac{\Gamma 2}{\Gamma^\frac{1}{4}} t^\frac{1}{4} + c_2 \left( \frac{\Gamma 3}{\Gamma^\frac{9}{4}} t^\frac{5}{4} - \frac{\Gamma 1}{\Gamma^\frac{1}{4}} t^\frac{3}{4} \right) - \left( c_0^2 + \frac{1}{4} c_2^2 - c_0 c_2 \right) \frac{\Gamma 1}{\Gamma^\frac{7}{4}} t^\frac{3}{4} - (2c_0 c_1 - c_1 c_2) \frac{\Gamma 2}{\Gamma^\frac{11}{4}} t^\frac{7}{4} - \left( 3c_0 c_2 - \frac{3}{2} c_2^2 + c_1^2 \right) \frac{\Gamma 3}{\Gamma^\frac{15}{4}} t^\frac{11}{4} - 3c_1 c_2^2 \frac{\Gamma 4}{\Gamma^\frac{19}{4}} t^\frac{15}{4} - \frac{9}{4} c_2^4 \frac{\Gamma 5}{\Gamma^\frac{23}{4}} t^\frac{19}{4} = f(t) \quad (26)
\]

and comparing coefficients of powers of \( t \) of equation (26), we have

\[
c_0 \frac{\Gamma 1}{\Gamma^\frac{1}{4}} = 0, \quad c_1 \frac{\Gamma 2}{\Gamma^\frac{1}{4}} - \frac{\Gamma 1}{\Gamma^\frac{1}{4}} \frac{\Gamma 3}{\Gamma^\frac{3}{4}} = 0, \quad c_2 \frac{\Gamma 3}{\Gamma^\frac{3}{4}} = 0, \quad \left( c_0^2 - \frac{1}{4} c_2^2 + c_0 c_2 \right) \frac{\Gamma 1}{\Gamma^\frac{3}{4}} = 0, \quad (2c_0 c_1 + c_1 c_2) \frac{\Gamma 2}{\Gamma^\frac{7}{4}} = 0, \quad \left( 3c_0 c_2 + \frac{3}{2} c_2^2 - c_1^2 \right) \frac{\Gamma 3}{\Gamma^\frac{11}{4}} = 0, \quad -3c_1 c_2 \frac{\Gamma 4}{\Gamma^\frac{15}{4}} = 0, \quad -\frac{9}{4} c_2^4 \frac{\Gamma 5}{\Gamma^\frac{19}{4}} = 0
\]

Solving the set of equations (27) we have

\[
c_0 = 0, \quad c_1 = 1, \quad c_2 = 0
\]

Substituting these values into the assumed solution, we have the approximate solution
This is the exact solution.

Example 3:

Consider the initial Value problem that consist of multi-fractional order non-linear integro differential equation

\[ D^{0.75}y(t) = \frac{1}{\Gamma(1.25)}t^{0.25} - \frac{2}{\Gamma(3.75)}t^{2.75} + J^{0.75}y^2(t) \quad t \in [0, 1] \]  

(29)

together with the initial condition; y(t) = 0

The exact solution is y(t) = t

We use power series method to solve this IVP of multi-order fractional problem N = 2 by assuming an approximate solution of the form:

Let \( y_2(t) = \sum_{i=0}^{2} a_i t^i = a_0 t^0 + a_1 t^1 + a_2 t^2 \)

(30)

\[ D^{0.75}y(t) = D^{0.75}y_2(t) = \frac{1}{\Gamma(1.25)}t^{0.75} + \frac{2}{\Gamma(3.75)}t^{2.25} \]

\[ = a_0 \frac{\Gamma_1}{\Gamma_4} t^{-\frac{1}{4}} + a_1 \frac{\Gamma_2}{\Gamma_4} t^\frac{1}{4} + a_2 \frac{\Gamma_3}{\Gamma_4} t^\frac{3}{4} + a_3 \frac{\Gamma_4}{\Gamma_4} t^{\frac{5}{4}} + a_4 \frac{\Gamma_5}{\Gamma_4} t^{\frac{7}{4}} \]

(31)

\[ y_2(t) = a_0 t^0 + a_1 t^1 + a_2 t^2 \]

\[ y_2^2(t) = a_0^2 + 2a_1 a_0 t + a_1^2 t^2 + 2a_0 a_2 t^2 + 2a_1 a_2 t^3 + a_2^2 t^4 \]

(32)

\[ J^{0.75}y^2(t) = a_0^2 \frac{\Gamma_1}{\Gamma_4} t^{\frac{1}{4}} + 2a_0 a_1 \frac{\Gamma_2}{\Gamma_4} t^{\frac{1}{4}} + (a_1^2 + 2a_0 a_2) \frac{\Gamma_3}{\Gamma_4} t^{\frac{3}{4}} + 2a_1 a_2 \frac{\Gamma_4}{\Gamma_4} t^{\frac{5}{4}} + a_2^2 \frac{\Gamma_5}{\Gamma_4} t^{\frac{7}{4}} \]

(33)
Now putting (31) and (33) in (29) and comparing coefficients of the powers of \( t \), we have

\[
a_0 = 0, \quad a_1 = 1, \quad a_2 = 0
\]

Substituting these values into (30), we have

\[
y_2(t) = t \quad \text{which is the exact solution.}
\]

**Example 4:**
Consider the initial value problem that consist of the multi-fractional order integro differential equation

\[
D^{0.5}y(t) = \frac{6}{\Gamma(3.5)} t^{2.5} - \frac{\Gamma(4)}{\Gamma(5.5)} t^{4.5} + J^{1.5}y(t) \quad t \in [0, 1]
\]

Subject to \( y(0) = 0 \). The exact solution is \( y(t) = t^3 \).

We use power series method to solve this IVP of multi-order fractional problem \( N = 4 \) by assuming an approximate solution in it’s form.

\[
y_4(t) = \sum_{i=0}^{4} a_i t^i = a_0 t^0 + a_1 t^1 + a_2 t^2 + a_3 t^3 + a_4 t^4
\]

Following the procedure of the method, we get the values of \( a_i \)’s,

\[
a_0 = 0, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 1, \quad a_4 = 0
\]

and so the approximate solution is

\[
y_4(t) = t^3
\]

which is the exact solution.

**Example 5**
Consider the initial value problem that consist of the multi-fractional order integro-
differential equation:

\[
D^\frac{5}{4} \varphi(t) = -\frac{\Gamma(5)}{\Gamma\left(\frac{15}{4}\right)} t^\frac{7}{4} + \frac{126}{\Gamma\left(\frac{19}{4}\right)} t^\frac{15}{4} + \frac{120}{\Gamma\left(\frac{27}{4}\right)} t^\frac{23}{4} + J^\frac{3}{4} \varphi(t), \ t \in [0, 1]
\]  \(35\)

Subject to \(\varphi(0) = 0\). Exact solution is \(\varphi(t) = t^5 - t^3\).

Using power series, we assume an approximate solution of the form

\[
\varphi(t) = \sum_{i=0}^{6} a_i t^i = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6
\]  \(36\)

substituting the above in the problem gives

\[
D^\frac{5}{4} \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6\} = \\
J^\frac{3}{4} \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6\}
\]

where \(f(t) = -\frac{\Gamma(5)}{\Gamma\left(\frac{15}{4}\right)} t^\frac{7}{4} + \frac{126}{\Gamma\left(\frac{19}{4}\right)} t^\frac{15}{4} + \frac{120}{\Gamma\left(\frac{27}{4}\right)} t^\frac{23}{4}\).

\[
\Rightarrow a_0 \frac{1}{\Gamma\left(\frac{1}{4}\right)} t^{-\frac{5}{4}} + a_1 \frac{\Gamma(2)}{\Gamma\left(\frac{3}{4}\right)} t^{-\frac{1}{4}} + a_2 \frac{\Gamma(3)}{\Gamma\left(\frac{7}{4}\right)} t^\frac{3}{4} + a_3 \frac{\Gamma(4)}{\Gamma\left(\frac{11}{4}\right)} t^\frac{7}{4} + a_4 \frac{\Gamma(5)}{\Gamma\left(\frac{15}{4}\right)} t^\frac{11}{4} + a_5 \frac{\Gamma(6)}{\Gamma\left(\frac{19}{4}\right)} t^\frac{15}{4} + a_6 \frac{\Gamma(7)}{\Gamma\left(\frac{23}{4}\right)} t^\frac{19}{4} \\
- \frac{\Gamma(5)}{\Gamma\left(\frac{15}{4}\right)} t^\frac{7}{4} + \frac{126}{\Gamma\left(\frac{19}{4}\right)} t^\frac{15}{4} + \frac{120}{\Gamma\left(\frac{27}{4}\right)} t^\frac{23}{4} + a_0 \frac{1}{\Gamma\left(\frac{1}{4}\right)} t^{-\frac{5}{4}} + a_1 \frac{\Gamma(2)}{\Gamma\left(\frac{3}{4}\right)} t^{-\frac{1}{4}} + a_2 \frac{\Gamma(3)}{\Gamma\left(\frac{7}{4}\right)} t^\frac{3}{4} + a_3 \frac{\Gamma(4)}{\Gamma\left(\frac{11}{4}\right)} t^\frac{7}{4} + a_4 \frac{\Gamma(5)}{\Gamma\left(\frac{15}{4}\right)} t^\frac{11}{4} + a_5 \frac{\Gamma(6)}{\Gamma\left(\frac{19}{4}\right)} t^\frac{15}{4} + a_6 \frac{\Gamma(7)}{\Gamma\left(\frac{23}{4}\right)} t^\frac{19}{4}
\]

\(37\)

Comparing coefficients, we have

\[
\begin{align*}
a_0 \frac{1}{\Gamma\left(\frac{1}{4}\right)} &= 0, & a_1 \frac{\Gamma(2)}{\Gamma\left(\frac{3}{4}\right)} &= 0, & a_2 \frac{\Gamma(3)}{\Gamma\left(\frac{7}{4}\right)} - a_0 \frac{1}{\Gamma\left(\frac{1}{4}\right)} &= 0, \\
a_3 \frac{\Gamma(4)}{\Gamma\left(\frac{19}{4}\right)} - a_1 \frac{\Gamma(2)}{\Gamma\left(\frac{3}{4}\right)} &= -\frac{\Gamma(4)}{\Gamma\left(\frac{19}{4}\right)}, & a_5 \frac{\Gamma(6)}{\Gamma\left(\frac{27}{4}\right)} - a_3 \frac{\Gamma(4)}{\Gamma\left(\frac{19}{4}\right)} &= \frac{126}{\Gamma\left(\frac{19}{4}\right)}, \\
a_6 \frac{\Gamma(7)}{\Gamma\left(\frac{31}{4}\right)} - a_4 \frac{\Gamma(5)}{\Gamma\left(\frac{23}{4}\right)} &= 0, & a_5 \frac{\Gamma(6)}{\Gamma\left(\frac{27}{4}\right)} &= \frac{120}{\Gamma\left(\frac{27}{4}\right)}, \\
& a_6 \frac{\Gamma(7)}{\Gamma\left(\frac{31}{4}\right)} &= 0
\end{align*}
\]  \(38\)
Solving the set of equations (38) for \(a_i\)s, we have
\[
a_0 = 0, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = -1, \quad a_4 = 0, \quad a_5 = 1, \quad a_6 = 0
\]
Substituting into the assumed approximate solution, we have
\[
\varphi_6(t) = t^5 - t^3, \quad \text{which is the exact solution.}
\]

5 Summary
Numerical examples to demonstrate the method was based on two main classes of problems; linear and Non-Linear problems. Basically, we solved three problems using the Power series and Legendre polynomial as bases functions. The results show that we obtain the exact solution for the Linear and Non-Linear problems. The proposed method is efficient, effective and has high convergence rate.

References