NUMERICAL SOLUTION OF $k^{th}$-ORDER LINEAR AND NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS COUPLED WITH LAPLACE TRANSFORM AND ADOMIAN DECOMPOSITION METHODS

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Abstract
In this work, the Laplace Transform Method is coupled with Adomian Decomposition Method to solve linear and nonlinear integro-differential equations. The nonlinear term is easily handled with the construction of Adomian Polynomials and laplace Transform is applied to the resulting linear problem and hence reduced the problem to its recursive relation and thus, leads to iterative procedure. The proposed method is illustrated with some examples that are both linear and nonlinear of varying orders. The results obtained with the proposed method compared favourably with existing results in literature and the exact solutions where such exists in closed form.

Keywords: Laplace Transform Method(LTM); Linear and Nonlinear Integro-Differential Equations; Addomian Decomposition Method(ADM); Adomian Polynomials.

1. Introduction
Integral and Integro-differential equations phenomena play a crucial role in many branches of linear and nonlinear Numerical Analysis and their applications are in the theory of Sciences, Physical, Social Sciences and Engineering problems. The results of solving linear and nonlinear equations can guide the authors to know the described process deeply. It is difficult for us to obtain the exact solution for these problems. More details[Wazwaz(2006)] and references are cited there in.
In recent decades, both mathematicians and physicists have devoted considerable effort to the study of numerical analysis[Adomian(1994) and Wazwaz(2011)] and the exact solution for linear and nonlinear integral and integro differential equations. The Adomian decomposition method has been shown to solve [Wazwaz(2002), Adomian(1994) and Jafari(2005, 2006)] efficiently, easily and accurately a large class of linear and nonlinear ordinary, partial and integro-differential equations. It has shown [Abboui and Cherrualt(1999), Hosseini and Nasabzadeh(2006)] that the series converges fast and with a few terms, this series approximate the exact solution with fairly reasonable error. Each term of this series gives generalized polynomial called the Adomian polynomials. Besides, the method does not require unnecessary linearization, perturbation, dicretization or unrealistic assumptions which may change the problem being solved.
The main objective of this work is to use the Combined Laplace Transform-Adomian Decomposition Method (CLT-ADM) in solving the $k^{th}$-order linear and nonlinear integro-differential equations. Let us consider the general functional equations.

\[ g(x, s) - Ng(x, s) = f \]  

where $N$ is a nonlinear operator, $f$ is a given function.

The Standard Adomian method defines the solution $g$ by the series;

\[ g(x, s) = \sum_{k=0}^{\infty} g_k(x, s) \]  

Nonlinear $N$ is decomposed as

\[ Ng(x, s) = \sum_{k=0}^{\infty} A_k(x, s) \]  

Putting Eq. (1), Eq. (2) into Eq. (3), we have

\[ \sum_{k=0}^{\infty} g_k - \sum_{k=0}^{\infty} A_k = f \]  

where $A_k$ are Adomian polynomials that can be constructed for various classes of nonlinearity according to specific algorithms set by Adomian (1992).

\[ A_0 = f(g_0) = g_0^3, \]
\[ A_1 = g_1 f'(g_0) = 2g_0 g_1, \]
\[ A_2 = g_2 f'(g_0) + 1/2! g_1 g_2 f''(g_0) = 2g_0 g_2 + g_1^2, \]
\[ A_3 = g_3 f''(g_0) + g_1 g_2 f''(g_0) + 1/3! f'''(g_0), \]
\[ \vdots \]
\[ A_k (g_0, g_1, g_2, \ldots, g_k) = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[ f\left( \sum_{j=0}^{\infty} \lambda^j g_j \right) \right]_{\lambda=0}, k \geq 0 \]  

where $\lambda$ is a parameter introduced for convenience, and $A_k, k \geq 0$ are polynomials known as Adomian polynomials and can be evaluated for all forms of nonlinearity.

1.1 DEFINITIONS OF SOME RELEVANT TERMS

**Definition 1.1.1:**

**Integro-Differential Equation:** an integro-differential equation is an equation in which the unknown $g(x)$ appears under the integral sign and contains an ordinary derivative $g^k$ as well. A standard integro-differential equation is of the form;

\[ g^k(x) = f(x) + \lambda \int_{i(x)}^{h(x)} k(x, s) g(s) ds \]

where $i(x)$ and $h(x)$ are the limits of integration respectively, which may be constants, variables or combined, $\lambda$ is a constant parameter, $f(x)$ is a given function and $k(x, s)$ is a known function of two variables $x$ and $s$, called the kernel.
Action of the unknown function \( g(x) \) inside the integral sign is one, the Integral or Integro-differential equation is called \textbf{linear} otherwise, it is called \textbf{nonlinear}.

\textbf{Definition 1.1.3}: Let \( g(s) \) be a function on the interval \([0, \infty)\), the \textbf{Laplace Transform} of \( g(s) \) is the function (in terms of \( z \)) given as:

\[
L\{g(s)\} = \int_0^\infty g(s)ds
\]  

Because the Laplace Transform yields a function of \( s \), we often use the notation:

\[
L\{g(s)\} = G(z)
\]

to denote the Laplace Transform of \( g(s) \).

\textbf{Definition 1.1.4}: The \textbf{inverse Laplace Transform} of a function \( G(z) \) is a unique continuos function \( g(s) \) on \([0, \infty)\) that satisfies \( L\{g(s)\} = G(z) \).

We denote the inverse of the Laplace Transform of \( G(z) \) as

\[
g(s) = L^{-1}\{G(z)\}
\]

\textbf{Definition 1.1.5}: \textbf{Taylor Series}: let \( g(x) \) be a function with derivatives of all orders in an interval \([x_0, x_1]\) that contains an interior point \( a \). The Taylor series of \( g(x) \) generated at \( x=a \) is

\[
g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(a)}{k!} (x-a)^k
\]

\textbf{Definition 1.1.6}: \textbf{Maclaurin Series}: the Taylor Series generated by \( g(x) \) at \( a=0 \) is called the Maclaurin Series and given by

\[
g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(a)}{k!} x^k
\]

\section{Methods of Solution of \( k \)-order Linear and Nonlinear Integro-Differential Equations by Combined Laplace Transform and Adomian Decomposition Methods.}

In this section, we discuss the use of Laplace Transform algorithm for the linear and nonlinear integro differential equations.

We consider the \( k \)-order Fredholm and Volterra integro-differential equations of the form.

\[
g^{(k)}(x) + f(x)g(x) + \int_{a(x)}^{b(x)} K(x, s)g^{(i)}(s)ds = h(x), i < x < l
\]

with initial conditions

\[
g(a) = \beta_0, g'(a) = \beta_1, g''(a) = \beta_2, \ldots, g^{(k-1)}(a) = \beta_{k-1}
\]
where $\beta_1 = 0, 1, \ldots, k - 1$ are real constants, $j$ and $k$ are integers and $j < k$.

The methodology consists of applying Laplace Transform first on both sides of Eq.(12)

$$L\left\{g^k(x)\right\} = z^k L\left\{g(x)\right\} - z^{k-1} g(0) - z^{k-2} g'(0) - \ldots - g^{(k-1)}(0)$$

(14)

Then,

$$z^k L\left\{g(x)\right\} - z^{k-1} g(0) - z^{k-2} g'(0) - \ldots - g^{(k-1)}(0)$$

(15)

$$= L\{h(x)\} - L\{f(x) * g(x)\} - L\left\{\int_{(x)}^{(s)} K(x,s) g^{(j)}(s) ds\right\}$$

(16)

$$= L\{h(x)\} - L\{f(x)\} L\left\{g(x)\right\} - \int_{(x)}^{(s)} L\{K(x,s) g^{(j)}(s) ds\}$$

(17)

By applying the properties of convolution(Polyanin and Mazhnox(1998); Jerri(1999)) on Eq.(17) gives:

$$L\left\{g(x)\right\} = \frac{z^{k-1} g(0) + z^{k-2} g'(0) + \ldots + g^{(k-1)}(0)}{z^k + L\{f(x)\}} + L\{h(x)\}$$

(18)

Putting Eq.(1), Eq.(2) and Eq.(3) into Eq.(18) we obtain:

$$L\left\{\sum_{k=0}^{\infty} g_k(x,s)\right\} = \frac{z^{k-1} g(0) + z^{k-2} g'(0) + \ldots + g^{(k-1)}(0)}{z^k + L\{f(x)\}} + L\{h(x)\}$$

(19)

and

$$= \frac{1}{z^k + L\{f(x)\}} L\left\{k(x,s)\right\} L\sum_{k=0}^{\infty} g_k(s) ds$$

(20)

In general the recursive relation is given by

$$L\left\{g_0(x)\right\} = \frac{z^{k-1} g(0) + z^{k-2} g'(0) + \ldots + g^{(k-1)}(0)}{z^k + L\{f(x)\}} + L\{h(x)\}$$

and

$$= \frac{1}{z^k + L\{f(x)\}} \int_{(x)}^{(s)} L\left\{k(x,s)\right\} g_k(s) ds, \quad k \geq 0$$

(21)

or
$$L\{g_0(x)\} = \frac{z^{-1}g(0)}{z + L\{f(x)\}} + \frac{L\{h(x)\}}{z + L\{f(x)\}}$$

and

$$L\{g_{k+1}(x)\} = -\frac{1}{z^k + L\{f(x)\}}L\{k(x,s)\}L\{A_k(x,s)\}, \quad k \geq 0 \quad (22)$$

A necessary condition for Eq.(21) and Eq.(22) to converge is that

$$\lim_{k \to \infty} \frac{1}{z^k + L\{f(x)\}} = 0 \quad (23)$$

We then define the $k$-terms approximant to the solution $g(x)$ by

$$\varphi_k\{g(x)\} = \sum_{i=1}^{k-1} g_i(x) \quad (24)$$

The obtained series solution converges to the exact solution.

3. Applications:

To illustrate this method for Linear and Nonlinear Integro-differential equations. We take five examples in this section.

Example 1

Consider the first-order linear Fredholm integro-differential equation by using the CLT-ADM [Alao et al. (2014)].

$$g'(x) = 1 - \frac{x}{3} + \int_0^x x s g(s) ds, \quad (26)$$

with initial condition

$$g(0) = 0 \quad (27)$$

Applying Laplace Transform to both sides of Eq.(26) and using the initial condition gives;

$$L\{g'(x)\} = L\left\{1 - \frac{x}{3}\right\} + L\left\{\int_0^x x s g(s) ds\right\} \quad (28)$$

$$z G(z) - g(0) = \frac{1}{z} - \frac{1}{3z^2} + \frac{1}{z} \int_0^x s g(s) ds \quad (29)$$

or

$$G(z) = \frac{1}{z^2} + \frac{1}{3z^3} + \frac{1}{z} \int_0^x s g(s) ds \quad (30)$$

Substituting the series assumption for $G(z)$ as given in Eq.(2) and using the recursive relation Eq.(21), we obtain;
\[ G(z) = \frac{1}{z^2} + \frac{1}{3z^3} \]

and

\[ L\{g_{k+1}(x)\} = \frac{1}{z^3} \int_0^z s g_k(s) ds, k \geq 0 \]  

(31)

Taking the inverse Laplace Transform of both sides of the first part of Eq.(31) gives \( g_0(x) \) and using the recursive relation Eq.(31) gives;

\[ g_0(x) = x - \frac{1}{3!} x^2 \]  

(32)

\[ \vdots \]

\[ g_k(x) = \frac{7x^2}{3!8^k}, k \geq 1 \]  

(33)

Thus, the series solution is given by

\[ \varphi_k(x) = \sum_{i=0}^{k-1} g_i(x) = x - \frac{1}{3!8^{k-1}} x^2, k \geq 1 \]  

(34)

\[ g(x) = \lim_{k \to \infty} \varphi_k(x) = \lim_{k \to \infty} \left\{ x - \frac{1}{3!8^{k-1}} x^2 \right\} = x \]  

(35)

that converges to the exact solution

\[ g(x) = x \]  

(36)

**Example 2**

Consider the first-order linear Fredholm integro-differential equation by using the CLT-ADM [Alao et al (2014)].

\[ g'(x) = xe^x + e^x - x + \int_0^1 (x-s) g(s) ds, \]  

(37)

with initial condition

\[ g(0) = 0 \]  

(38)

Applying Laplace Transform to both sides of Eq.(37) and using the initial conditions gives;

\[ L\{g'(x)\} = L\{xe^x + e^x - x\} + L\left\{ \int_0^1 (x-s) g(s) ds \right\} \]  

(39)

\[ zG(z) - g(0) = \frac{1}{(z-1)^2} + \frac{1}{z-1} - \frac{1}{z^2} + \frac{1}{z^3} \int_0^1 g(s) ds \]  

(40)

\[ G(z) = \frac{1}{z(z-1)^2} + \frac{1}{z(z-1)} - \frac{1}{z^2} + \frac{1}{z^3} \int_0^1 g(s) ds \]  

(41)

Substituting the series assumption for \( G(z) \) as given in Eq.(2) and using the recursive relation Eq.(21) we obtain;
\[ G(z) = \frac{1}{z(z-1)^2} + \frac{1}{z(z-1)} - \frac{1}{z^3} \]

and

\[ L\{g_{k+1}(x)\} = \frac{1}{z^i} \int_0^1 g_k(s)ds, k \geq 0 \quad (42) \]

Taking the inverse Laplace Transform of both sides of the first part of Eq.(42) gives \( g_0(x) \) and using the recursive relation Eq.(42) gives;

\[ g_0(x) = xe^x - \frac{1}{2}x^2, \quad (43) \]

when \( k = 0 \)

\[ L\{g_1(x)\} = \frac{1}{z^i} \int_0^1 g_0(s)ds \quad (44) \]

\[ g_1(z) = \frac{1}{z^i} \int_0^1 \left( se^s - \frac{1}{2}s^2 \right)ds \quad (45) \]

\[ g_1(z) = \frac{1}{z^i} \left\{ \frac{5}{6} \right\} \quad (46) \]

\[ L^{-1}\{g_1(z)\} = L^{-1}\left\{ \frac{1}{z^i} \cdot \frac{5}{6} \right\} \quad (47) \]

\[ g_1(x) = \frac{5x^2}{12} \quad (48) \]

\[ \vdots \quad (49) \]

\[ g_k(x) = \frac{5x^2}{2!6^i}, k \geq 1 \]

Thus, the series solution is given by

\[ \varphi_k(x) = \sum_{i=0}^{k-1} g_i(x) = xe^x - \frac{1}{2!6^k}x^2, k \geq 1 \quad (50) \]

\[ g(x) = \lim_{k \to \infty} \varphi_k(x) = \lim_{k \to \infty} \left\{ xe^x - \frac{1}{2!6^k}x^2 \right\} = xe^x \quad (51) \]

that converges to the exact solution

\[ g(x) = xe^x \quad (52) \]
Consider the first-order nonlinear Fredholm integro-differential equation by using the CLT-ADM [Wazwaz (2011)].

\[ g'(x) = \sin(x) - \frac{\pi}{80} + \frac{1}{120} \int_{0}^{\pi} x g^2(s) ds, \quad (53) \]

with initial condition

\[ g(0) = 0 \quad (54) \]

Applying Laplace Transform of both sides of Eq.(53) and using the initial condition we have;

\[ L\{g'(x)\} = L\left\{ \sin(x) - \frac{\pi}{80} + \frac{1}{120} \int_{0}^{\pi} x g^2(s) ds \right\} \quad (55) \]

\[ zG(z) - g(0) = -\frac{1}{z^2+1} + \frac{\pi}{80z} + \frac{1}{120z^2} \int_{0}^{\pi} g^2(s) ds \quad (56) \]

\[ zG(z) = \frac{1}{z^2+1} + \frac{\pi}{80z} + \frac{1}{120z^2} \int_{0}^{\pi} g^2(s) ds \quad (57) \]

or

\[ G(z) = \frac{1}{z^2+1} + \frac{\pi}{80z} + \frac{1}{120z^2} \int_{0}^{\pi} g^2(s) ds \quad (58) \]

Substituting the series assumption for \( G(z) \) as given in Eq.(2) and using the recursive relation Eq.(21) we obtain;

\[ G_0(z) = \frac{1}{z(z^2+1)} - \frac{\pi}{80z^2}, \]

and

\[ L\{g_{k+1}(x)\} = \frac{1}{120z^2} + \int_{0}^{\pi} A_k(s) ds, k \geq 0 \quad (59) \]

where \( A_k = \sum_{j=0}^{k} g_j g_{k-j}, k \geq 1, A_0 = 0 \)

Taking the inverse Laplace Transform of both sides of the first part of Eq.(59) gives \( g_0(x) \), and using the recursive relation Eq.(59) gives;

\[ g_0(x) = 1 - \cos(x) - \frac{\pi x}{80}, \quad (60) \]

when \( k=0 \)

\[ L\{g_1(x)\} = \frac{1}{120z^3} \int_{0}^{\pi} A_k(s) ds \quad (61) \]

\[ L\{g_1(x)\} = \frac{1}{120z^3} \int_{0}^{\pi} \left(1 - \cos(s) - \frac{\pi s}{80}\right) ds \quad (62) \]
\[ g_1(z) = \frac{1}{120z^3} \left( \frac{29\pi}{20} - \frac{\pi^3}{80} + \frac{\pi^5}{19200} \right) \] (63)

\[ L^{-1}\{g_1(z)\} = L^{-1}\left\{ \frac{29\pi}{2400z^3} - \frac{\pi^3}{9600z^3} + \frac{\pi^5}{2304000} \right\} \] (64)

\[ g_1(x) = \frac{29\pi x^2}{4800} - \frac{\pi^3 x^2}{19200} + \frac{\pi^5 x^2}{4608000} \] (65)

The series solution is given as

\[ g(x) = 1 - \cos x - \frac{\pi x}{80} + \frac{29\pi x^2}{4800} - \frac{\pi^3 x^2}{19200} + \frac{\pi^5 x^2}{4608000} \] (66)

That converges to the exact solution

\[ g(x) = 1 - \cos x \] (67)

**Example 4**

Consider the Second-order linear Volterra integro-differential equation by using the CLT-ADM [wazwaz (2011)].

\[ g''(x) = 1 + x + \int_0^x (x-s)g(s)ds, \] (68)

with initial conditions

\[ g(0) = 1, g'(0) = 1 \] (69)

Applying Laplace Transform to both sides of Eq.(68) and using the initial condition gives;

\[ L\{g'(z)\} = L\{1 + x\} + L\left\{ \int_0^x (x-s)g(s)ds \right\} \] (70)

\[ z^2G(z) - zg(0) - g'(0) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^2} L\{g(x)\} \] (71)

\[ z^2G(z) - z - 1 = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^2} L\{g(x)\} \] (72)

\[ z^2G(z) = z + 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^2} L\{g(x)\} \] (73)
\[ G(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^4} L\{g(x)\} \]  
(74)

Substituting the series assumption for \( G(z) \) as given in Eq.(2) and using the recursive relation Eq.(21) we obtain;

\[ G_0(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}, \]

and

\[ L\{g_{k+1}(x)\} = \frac{1}{z^4} L\{g_k(x)\} \]  
(75)

Taking the inverse Laplace transform of both sides of the first part of Eq.(75) gives \( g_0(x) \), and using the recursive relation Eq.(75) gives;

\[ g_0(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \]

\[ : \]

\[ g_k(x) = 1 + \frac{x^k}{k!}, k \geq 1 \]  
(77)

Thus, the series solution is given by

\[ \phi_k(x) = \sum_{k=0}^{k-1} \frac{x^k}{k!}, k \geq 0 \]  
(78)

that converges to the exact solution

\[ g(x) = e^x \]  
(79)

**Example 5**

Consider the first-order nonlinear Volterra integro-differential equation by using the CLT-ADM [Wafa and Fawzi (2014)].

\[ g'(x) = \frac{17}{4} + \frac{9}{2} x - 2x^2 - 3e^x - \frac{1}{4} e^{2x} + \int_0^x (x-s) g^2(s) ds, \]  
(80)

with initial condition

\[ g(0) = 3 \]  
(81)

Applying Laplace Transform to both sides of Eq.(80) and using the initial condition, we obtain;

\[ L\{g'(x)\} = L\left\{ \frac{17}{4} + \frac{9}{2} x - 2x^2 - 3e^x - \frac{1}{4} e^{2x} + \int_0^x (x-s) g^2(s) ds, \right\}, \]  
(82)

\[ zG(z) - g(0) = \frac{17}{4z} + \frac{9}{2z^2} - \frac{4}{z^3} - \frac{3}{z-1} - \frac{1}{4(z-2)} + \frac{1}{z^2} L\{g^2(x)\} \]  
(83)

or
\[ G(z) = \frac{3}{z} + \frac{17}{4z^2} + \frac{9}{2z^3} - \frac{4}{z^4} = \frac{3}{z(z-1)} - \frac{1}{4z(z-2)} + \frac{1}{z^3} L\{g^2(x)\} \]  

Substituting the series assumption for \( G(z) \) and the Adomian Polynomials for \( g^2(x) \) as given in Eq.(2) and Eq.(3) respectively, and using the recursive relation Eq.(22) gives;

\[ G_0(z) = \frac{3}{z} + \frac{17}{4z^2} + \frac{9}{2z^3} - \frac{4}{z^4} = \frac{3}{z(z-1)} - \frac{1}{4z(z-2)}, \]

and

\[ L\{g_{k+1}(x)\} = \frac{1}{z^3} L\{A_k(x)\}, k \geq 0 \]  

Recall that the Adomian polynomials for \( F(g(x)) = g^2(x) \); are given by

\[ A_0 = g_0^2, \]
\[ A_1 = 2g_0g_1 \]

Taking the inverse laplace Transform of both sides of the first part of Eq.(85) and using the recursive relation Eq.(85) gives;

\[ g_0(x) = 3 + x + \frac{x^2}{2} - \frac{4x^3}{3} - \frac{5x^4}{24} - \frac{7x^5}{120} + \ldots, \]  

\[ g_1(x) = \frac{3x^3}{2} + \frac{x^4}{4} + \frac{x^5}{15} + \frac{7x^6}{120} - \frac{11x^7}{630} - \frac{x^8}{160} + \ldots \]  

\[ + \frac{523x^9}{181440} + \frac{179x^{10}}{259200} + \frac{191x^{11}}{950400} + \frac{7x^{12}}{380160} - \frac{49x^{13}}{24710400} \]  

Using Eq.(86) and Eq.(87); the series solution is therefore given by

\[ g(x) = 3 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \]  

\[ g(x) = 2 + e^x \]  

4. CONCLUSION:

The main aim of this work, was to give an efficient simple method for solving both linear and nonlinear integro-differential equations. We successfully applied combined Laplace Transform-Adomian Decomposition Method (CLT-ADM). The main advantage of this method is the fact it gives the analytical solution. In the above examples we observed that the CLT-ADM with the initial approximation obtained from the initial conditions yield a good approximation to the exact solution only in a few iterations. It is shown that the method is a promising tool for solving linear and nonlinear integro-differential equations.
References

